Warsaw University of Technology
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

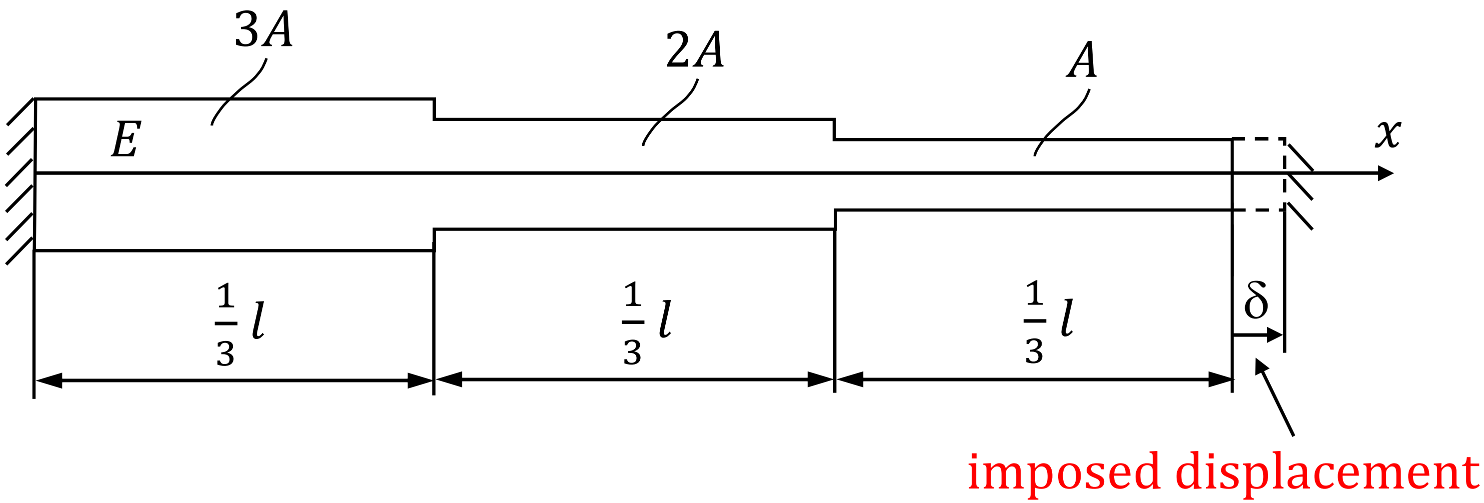
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

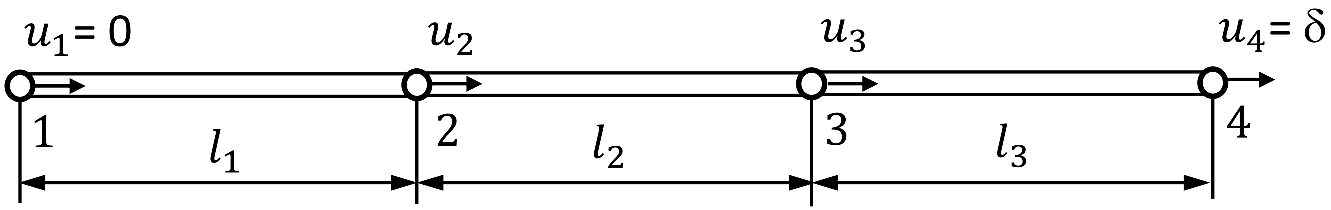
Imposed displacement and applied force

04.2021

FE model of a bar with imposed displacement

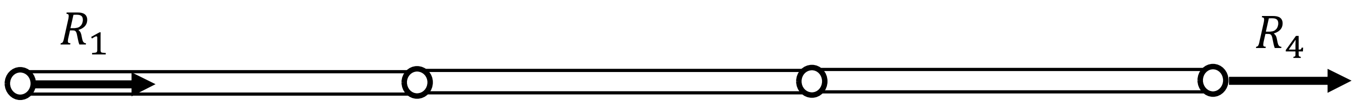


FE model – 3 bar elements:



$$\underset{1\times 4}{\lfloor q \rfloor} = \lfloor u_1, u_2, u_3, u_4 \rfloor$$

FBD:



$$\underset{1\times 4}{\lfloor F \rfloor} = \lfloor R_1, 0, 0, R_4 \rfloor \qquad (R_1 + R_4 = 0)$$

FE model of a bar with imposed displacement

stiffness matrices:

$$\underset{2\times 2}{[k]_e} = \frac{EA_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underset{2\times 2}{[k]_1} = \frac{E\cdot 3A}{\frac{1}{3}\,l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l}\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\underset{2\times 2}{[k]_2} = \frac{E\cdot 2A}{\frac{1}{3}\,l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l}\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}$$

$$\underset{2\times 2}{[k]_3} = \frac{EA}{\frac{1}{3}\,l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{EA}{l}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\underset{4\times 4}{[k]_1^*} = \frac{EA}{l}\begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underset{4\times 4}{[k]_2^*} = \frac{EA}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{4\times 4}{[k]_3^*} = \frac{EA}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \quad \rightarrow \quad \underset{4\times 4}{[K]} = \sum_{e=1}^{3} [k]_e^* = \frac{EA}{l}\begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

FE model of a bar with imposed displacement

set of equations: $\underset{4\times 4}{[K]} \cdot \underset{4\times 1}{\{q\}} = \underset{4\times 1}{\{F\}}$

$$\underset{4\times 4}{[K]} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underset{4\times 4}{[K]} \cdot \left(\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} \right) = \underset{4\times 4}{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} + \underset{4\times 4}{[K]} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} = \underset{4\times 1}{\{F\}}$$

$$\underset{4\times 4}{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \underset{4\times 1}{\{F\}} - [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} = \underset{4\times 1}{\{F\}} - \frac{EA}{l} \begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{Bmatrix} =$$

$$= \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix} - \frac{EA}{l} \begin{Bmatrix} 0 \\ 0 \\ -3\delta \\ 3\delta \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l}\delta \\ R_4 - \frac{3EA}{l}\delta \end{Bmatrix} \rightarrow \underset{4\times 4}{[K]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l}\delta \\ R_4 - \frac{3EA}{l}\delta \end{Bmatrix}$$

FE model of a bar with imposed displacement

boundary conditions: $u_1 = 0$

$$\frac{EA}{l}\begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ \frac{3EA}{l}\delta \\ R_4 - \frac{3EA}{l}\delta \end{Bmatrix}$$

u_2 , u_3 - unknown nodal parameters

$$\frac{EA}{l}\begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{3EA}{l}\delta \end{Bmatrix}$$

$$\begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\delta \end{Bmatrix}$$

$$\det\begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix} = 15 \cdot 9 - (-6)(-6) = 99 \quad ; \begin{bmatrix} 15 & -6 \\ -6 & 9 \end{bmatrix}^{CT} = \begin{bmatrix} 9 & 6 \\ 6 & 15 \end{bmatrix}$$

FE model of a bar with imposed displacement

displacements:

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{99}\begin{bmatrix} 9 & 6 \\ 6 & 15 \end{bmatrix}\begin{Bmatrix} 0 \\ 3\delta \end{Bmatrix}$$

$$u_2 = \frac{1}{99}(9 \cdot 0 + 6 \cdot 3\delta) = \frac{18\delta}{99} = \frac{2\delta}{11}$$

$$u_3 = \frac{1}{99}(6 \cdot 0 + 15 \cdot 3\delta) = \frac{45\delta}{99} = \frac{5\delta}{11}$$

$$N_1(\xi) = 1 - \frac{\xi}{l_e} = 1 - \frac{3\xi}{l}; \qquad N_2(\xi) = \frac{\xi}{l_e} = \frac{3\xi}{l}$$

$$u(\xi) = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e \quad \rightarrow \quad \varepsilon_x = \frac{du}{d\xi} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \left\lfloor -\frac{3}{l}, \frac{3}{l} \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}_1 \quad ; \quad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 \quad ; \quad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_3 = \begin{Bmatrix} u_3 \\ \delta \end{Bmatrix}_3$$

FE model of a bar with imposed displacement

strain in elements:

$$\varepsilon_{x1} = \left\lfloor -\frac{3}{l}, \frac{3}{l} \right\rfloor \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix}_1 = -\frac{3}{l} \cdot 0 + \frac{3}{l} \cdot u_2 = \frac{6\delta}{11l}$$

$$\varepsilon_{x2} = \left\lfloor -\frac{3}{l}, \frac{3}{l} \right\rfloor \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_2 = -\frac{3}{l} \cdot u_2 + \frac{3}{l} \cdot u_3 = \frac{9\delta}{11l}$$

$$\varepsilon_{x3} = \left\lfloor -\frac{3}{l}, \frac{3}{l} \right\rfloor \begin{Bmatrix} u_3 \\ \delta \end{Bmatrix}_3 = -\frac{3}{l} \cdot u_3 + \frac{3}{l} \cdot \delta = \frac{18\delta}{11l}$$

stress in elements:

$$\sigma_{x1} = E \cdot \varepsilon_{x1} = \frac{6E\delta}{11l}$$

$$\sigma_{x2} = E \cdot \varepsilon_{x2} = \frac{9E\delta}{11l}$$

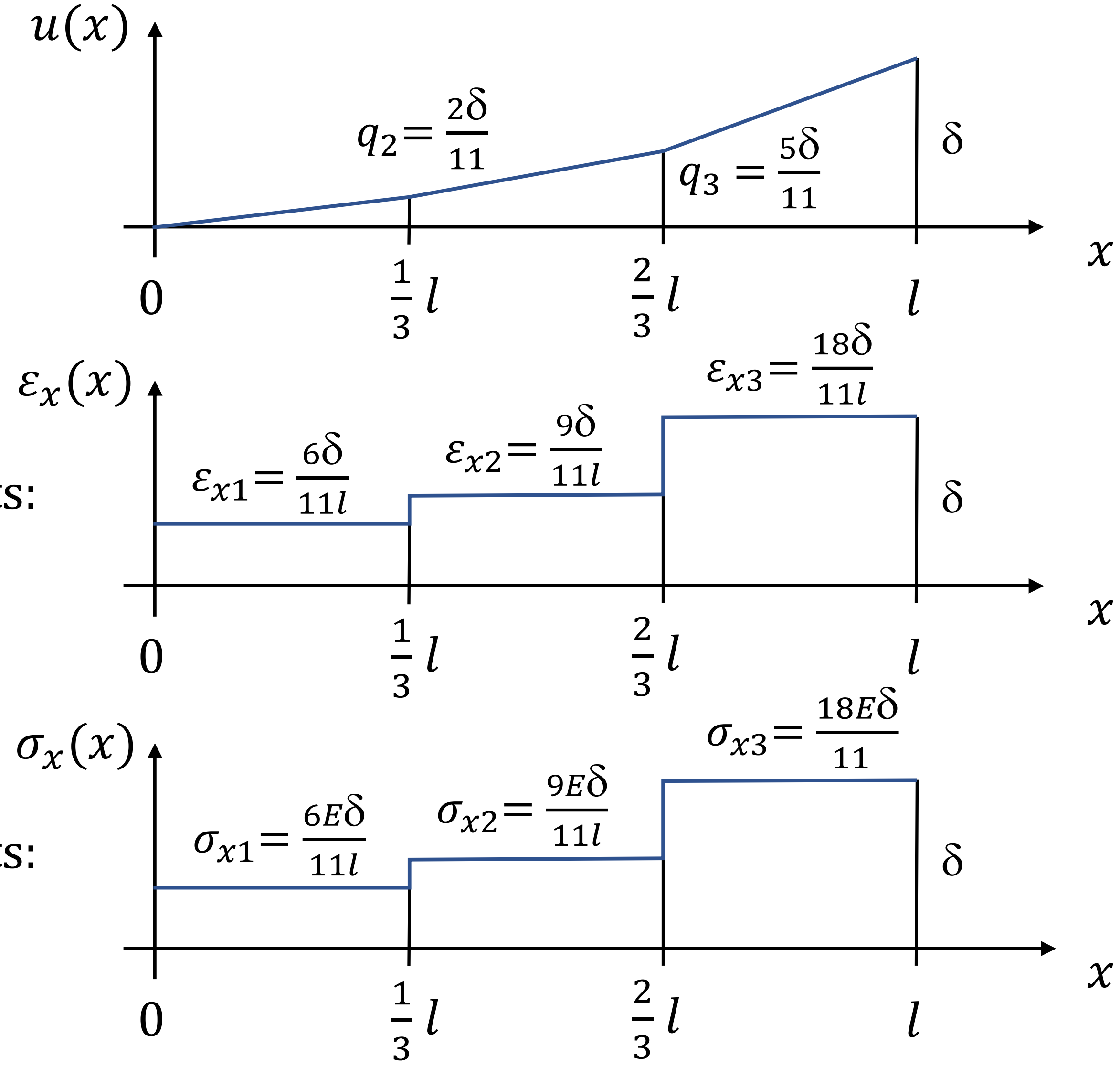
$$\sigma_{x3} = E \cdot \varepsilon_{x3} = \frac{18E\delta}{11l}$$

FE model of a bar with imposed displacement

DOF solution:

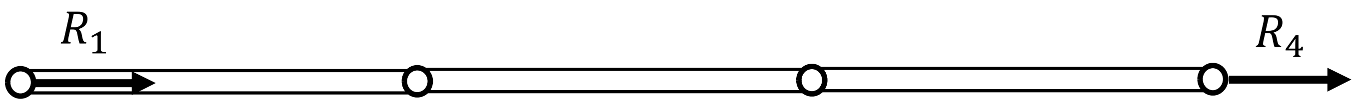
Strain in elements:

Stress in elements:



FE model of a bar with imposed displacement

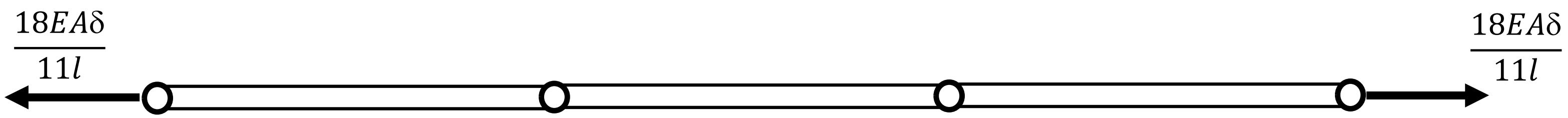
reactions:



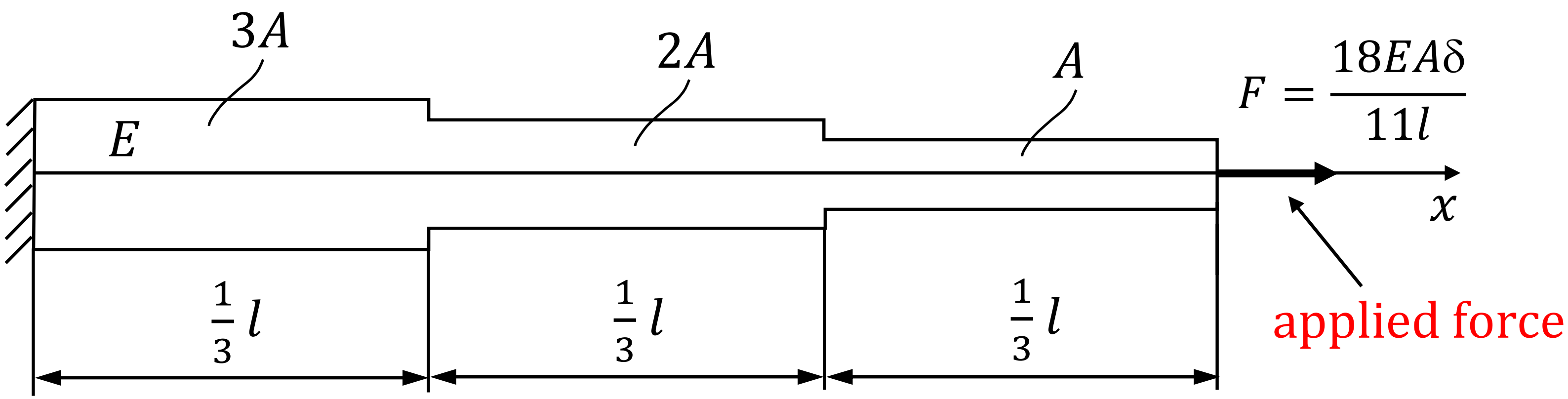
$$\frac{EA}{l}\begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{2\delta}{11} \\ \frac{5\delta}{11} \\ \delta \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix}$$

$$\frac{EA}{l}\left(9\cdot 0 - 9\cdot\frac{2\delta}{11} + 0\cdot\frac{5\delta}{11} + 0\cdot\delta\right) = R_1 \quad \rightarrow \quad R_1 = -\frac{18EA\delta}{11l}$$

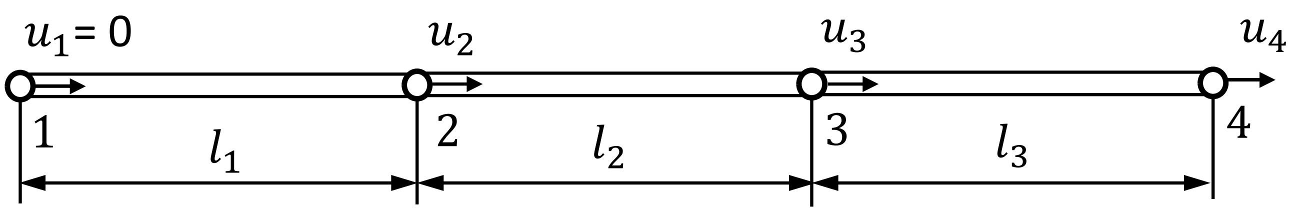
$$\frac{EA}{l}\left(0\cdot 0 + 0\cdot\frac{2\delta}{11} - 3\cdot\frac{5\delta}{11} + 3\cdot\delta\right) = R_4 \quad \rightarrow \quad R_4 = \frac{18EA\delta}{11}$$



FE model of a bar with applied force

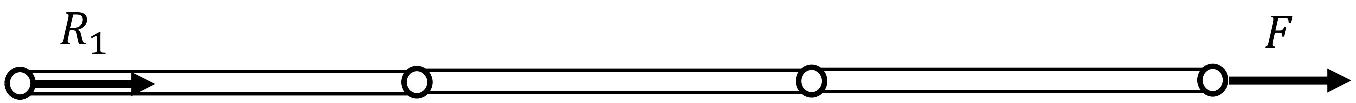


FE model – 3 bar elements:



$$\underset{1\times 4}{\lfloor q \rfloor} = \lfloor u_1, u_2, u_3, u_4 \rfloor$$

FBD:



$$\underset{1\times 4}{\lfloor F \rfloor} = \lfloor R_1, 0,0, F \rfloor \qquad (R_1 + F = 0)$$

FE model of a bar with applied force

boundary conditions: $u_1 = 0$

$$\frac{EA}{l}\begin{bmatrix} 9 & -9 & 0 & 0 \\ -9 & 15 & -6 & 0 \\ 0 & -6 & 9 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ F \end{Bmatrix}$$

u_2 , u_3 , u_4 - unknown nodal parameters

$$\frac{EA}{l}\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{18EA\delta}{11l} \end{Bmatrix}$$

$$\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

FE model of a bar with applied force

$$\det\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix} =$$

$$= 15 \cdot (9 \cdot 3 - (-3)(-3)) - (-6) \cdot \big((-6) \cdot 3 - (-3) \cdot 0\big) = 162$$

$$\begin{bmatrix} 15 & -6 & 0 \\ -6 & 9 & -3 \\ 0 & -3 & 3 \end{bmatrix}^{CT} = \begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{1}{162}\begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

FE model of a bar with applied force

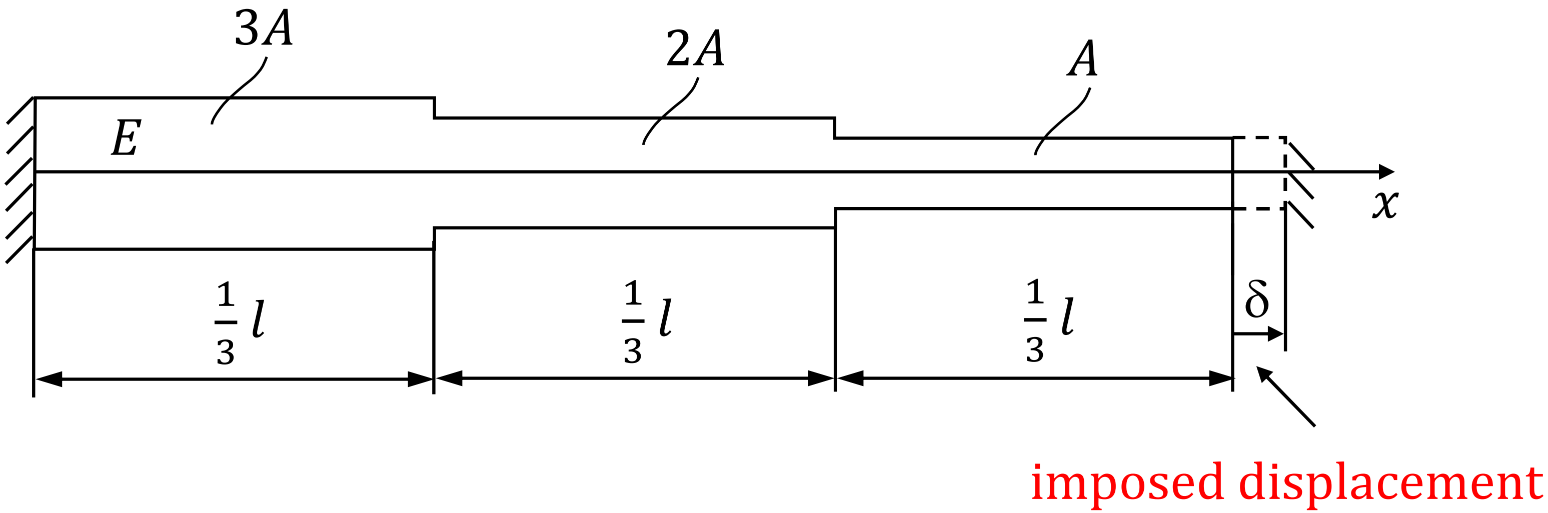
$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{1}{162}\begin{bmatrix} 18 & 18 & 18 \\ 18 & 45 & 45 \\ 18 & 45 & 99 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{18\delta}{11} \end{Bmatrix}$$

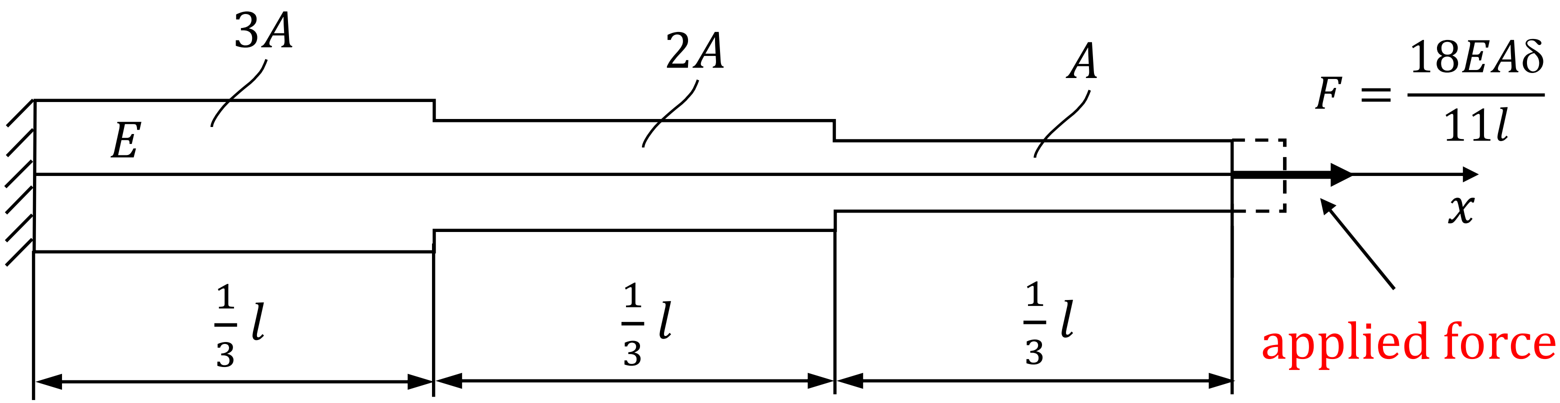
$$u_2 = \frac{1}{162}(18 \cdot 0 + 18 \cdot 0 + 18 \cdot \frac{18\delta}{11}) = \frac{18{\cdot}18\ \delta}{162{\cdot}11} = \frac{18{\cdot}9{\cdot}\ \ \delta}{18{\cdot}9{\cdot}1} = \frac{2\delta}{11}$$

$$u_3 = \frac{1}{162}(18 \cdot 0 + 45 \cdot 0 + 45 \cdot \frac{18\delta}{11}) = \frac{45{\cdot}1\ \ \delta}{162{\cdot}11} = \frac{5{\cdot}9{\cdot}18\ \delta}{18{\cdot}9{\cdot}1} = \frac{5\delta}{11}$$

$$u_4 = \frac{1}{162}(18 \cdot 0 + 45 \cdot 0 + 99 \cdot \frac{18\delta}{11}) = \frac{99{\cdot}18\ \delta}{162{\cdot}11} = \frac{11{\cdot}9{\cdot}18\ \delta}{18{\cdot}9{\cdot}1} = \frac{11\delta}{11} = \delta$$

Two types of load giving the same result:





Comparison between imposed displacement and force

Load type	Set of FE equations	N
Imposed displacement δ	$\frac{EA}{l}\begin{bmatrix}15 & -6\\ -6 & 9\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0\\ \frac{3EA}{l}\delta\end{Bmatrix}$ $cond\left(\begin{bmatrix}15 & -6\\ -6 & 9\end{bmatrix}\right)=4.45$	2
Force $F=\frac{18EA\delta}{11l}$	$\frac{EA}{l}\begin{bmatrix}15 & -6 & 0\\ -6 & 9 & -3\\ 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ \frac{18EA\delta}{11l}\end{Bmatrix}$ $cond\left(\begin{bmatrix}15 & -6 & 0\\ -6 & 9 & -3\\ 0 & -3 & 3\end{bmatrix}\right)=21$	3

Condition number $cond([K]) \approx 1$ – problem well conditioned, $(cond) \gg 1$ – ill conditioned

$$cond([K]) = \|K\|_\infty \cdot \|K^{-1}\|_\infty$$